CATEGORY THEORY
CATEGORY IV - METRIC SPACES

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Definition 1 (Objects)**.** Let X be a set. A *metric* on X is a function

$$d : X \times X \to \mathbb{R}$$

satisfying

(M1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (Positivity);
(M2) $d(x, y) = d(y, x)$ (Symmetry);
(M3) $d(x, y) + d(y, z) \geq d(x, z)$ (Triangle Inequality).

The pair (X, ρ) is called a *metric space*.

Definition 2 (Subobjects)**.** Let (X, d) be a metric space and let $Y \subset X$. If we restrict d to $Y \times Y$, we obtain a metric on Y. Then (Y, d) is a metric space, and is called a *subspace* of (X, d).

There are two distinct categories which we wish to investigate with respect to metric spaces, one in which the morphisms are isometries, and one in which the morphisms are continuous functions.

Definition 3 (Morphisms)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$. We say that f is *distance preserving*, or is an *isometry*, if

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

The identity map is an isometry, and the composition of isometries is an isometry. Thus, metric spaces together with isometries from a category.

Example 1. The set of real numbers is a metric space. The distance from x to y is defined by $\rho(x, y) = |x - y|$.

Example 2. Let $X = \mathbb{R}^2$ and use the Pythagorean theorem to define the metric ρ by

$$\rho(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

where $p = (x_1, y_1)$ and $q = (x_2, y_2)$.

Example 3. Let $X = \mathbb{R}^3$. Two applications of the Pythagorean theorem and some slight simplification leads to the definition of the metric ρ by

$$\rho(p, q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

where $p = (x_1, y_1, z_1)$ and $q = (x_2, y_2, z_2)$.

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Example 4. Let $X = \mathbb{R}^n$. We need to slightly modify our notation to conveniently write the distance formula. Thus for $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$, define

$$\rho(p, q) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}.$$

Any subset of $\mathbb{R}^n$ is a metric space by restriction of the metric to the subset. The *group of symmetries* of $X \subset \mathbb{R}^n$ is $\mathrm{Aut}(X)$, in the category of metric spaces with isometries as the morphisms.

Problem 1. Consider each of the following sets X as a metric space. Let $\mathrm{Aut}(X)$ be the group of isometries $X \to X$. Describe $\mathrm{Aut}(X)$ in each case.

- **(a)** $X = \{(-1, 0), (1, 0), (0, 1)\} \subset \mathbb{R}^2$
- **(b)** $X = \{(-1, 0), (1, 0), (0, \sqrt{3}\} \subset \mathbb{R}^2$
- **(c)** $X = \{(-1, 0), (1, 0), (0, 1), (0, -1)\} \subset \mathbb{R}^2$
- **(d)** $X = \mathbb{Z} \subset \mathbb{R}$
- **(e)** $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
- **(f)** $X = \{(x, y) \in \mathbb{R}^2 \mid 4x^4 + 9x^2 = 36\}$
- **(g)** $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

Problem 2. In each case, let $X = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}$. Then X is a subspace of the metric space $\mathbb{R}^2$, with isometries as morphisms. Describe $\mathrm{Aut}(X)$ in each case.

- **(a)** $f(x) = x^2$
- **(b)** $f(x) = x - x^3$
- **(c)** $f(x) = \dfrac{1}{x}$
- **(d)** $f(x) = \tan x$

Problem 3. Let $X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$.

- **(a)** Show that the points in X form the vertices of a regular tetrahedron.
- **(b)** Find $\mathrm{Aut}(X)$.

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